

COMPREHENSIVE EXAMINATION

Department of Physics
University of Nevada, Reno
January 12, 2004; 2:30-5:00 PM

CLASSICAL MECHANICS

Answer any four problems. Do not turn in solutions for more than four problems.

(1) The equation of motion along a straight line is...

(a)

$$m \frac{dv}{dt} = -av$$

with initial conditions $x(t=0) = 0$, $v(t=0) = v_0$. Also $v = \frac{dx}{dt}$. Determine $v = v(t)$ and $x(t)$. What is $\lim_{t \rightarrow \infty} v(\infty)$?

(b)

$$m \frac{dv}{dt} = -\frac{a}{v^2}$$

with initial conditions $x(t=0) = 0$, $v(t=0) = v_0$. Also $v = \frac{dx}{dt}$. Determine $v = v(t)$ and $x(t)$. What is $\lim_{t \rightarrow \infty} v(\infty)$?

(2) Derive the equations of Kepler's motion in the form of integrals $t = t(r)$, $\phi = \phi(r)$ without carrying out the integration.

(3) Which force $\vec{F} = \{X, Y, Z\}$ (where X, Y and Z are the Cartesian components of \vec{F}) has a potential and, if so, why?

(a) $X = ay$, $Y = 0$ and $Z = 0$.

(b) $X = f(x)$, $Y = 0$ and $Z = 0$.

(4) Consider the central elastic collision of two masses m_1 and m_2 , moving towards each other with some velocity. Question: Which equations are needed to compute the velocities of the masses following their collision, if their velocities before the collision are known?

(5) (a) Show how Lagrange's equation of motion of the second kind is obtained from Hamilton's principle.

(b) Show how Hamilton's equations of motion are obtained from Lagrange's equation of motion of the second kind.

Physics Comprehensive Exam, Spring 2004
Statistical Mechanics

Please solve four of the six problems below

1. The Maxwell-Boltzmann speed distribution for an ideal gas is given by

$$f(v)dv = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT} dv$$

where the number of particles with speeds between v and $v + dv$ is $Nf(v)dv$. Use this distribution to find expressions for the following average speed quantities as functions of the particle mass m and the temperature T .

- (a) The most probable speed v_{mp} (the maximum in the speed distribution).
- (b) The average speed, $\langle v \rangle$.
- (c) The root-mean-square speed, $\langle v^2 \rangle^{1/2}$.

2. Suppose that two systems are identical in every respect except that the energy levels of one have each been shifted by an arbitrary amount Δ relative to the other. As a result, the partition functions for the two systems can be generally written,

$$Q = \sum_j e^{-\beta E_j}$$

$$Q' = \sum_j e^{-\beta(E_j + \Delta)}$$

- (a) Show that both the average energy \bar{E} and the Helmholtz free energy of the primed system are shifted by Δ relative to the unprimed system.
 - (b) Show that the entropy, S , is the same for both systems.
 - (c) Show that the chemical potential μ and the pressure p do not depend on the value of Δ .
3. Consider the molecular partition function of an arbitrary diatomic molecule. Suppose that the vibrational partition function can be adequately approximated by the harmonic oscillator partition function. In this case, the thermal population of the n^{th} state can be written $P_n = \frac{1}{q_{\text{vib}}} e^{-\beta \varepsilon(n)}$ where $\varepsilon(n)$ gives the harmonic oscillator energy levels as a function of the quantum number n and

$$q_{\text{vib}} = \frac{e^{-\beta \hbar \omega / 2}}{1 - e^{-\beta \hbar \omega}}$$

- (a) Find an analytic expression for the temperature at which the n^{th} state has its maximum relative population (*i.e.* optimize $P_n(T)$). (8 points)
 - (b) Using the expression obtained in part (i), show that the largest population thermally attainable in the $n = 1$ state of a harmonic oscillator is $\frac{1}{4}$. Find a general formula for the maximum population of the n^{th} state.
4. The number of single point defects of the Schottky-type in a typical crystal is given by

$$n \approx \frac{Ne^{-\varepsilon_d/kT}}{1 + e^{-\varepsilon_d/kT}}$$

where ε_d is the energy required to displace an atom to the surface of the crystal.

- (a) Show that the number of pair defects of the Schottky type in an ionic crystal is given by

$$n = \frac{Ne^{-\varepsilon_p/2kT} e^{\Delta S_{vib}/k}}{1 + e^{-\varepsilon_p/2kT} e^{\Delta S_{vib}/k}}$$

where ε_p is the energy necessary to create a pair defect and ΔS_{vib} is the vibrational contribution to the entropy change induced by the pair-defect formation.

- (b) If there are N atoms in a crystal and a total of N' interstitial sites to which the atoms can be dislocated, show that the number of Frenkel defects can be written

$$n \approx \sqrt{NN'} e^{-\varepsilon_i/2kT} e^{\Delta S_{vib}/2k}$$

where ε_i is the energy required to move an atom from a lattice to an interstitial position. (Note: For pair defects,

$$\Delta S_{conf} = k \ln \left[\frac{N!}{n!(N-n)!} \right]^2 = 2k \ln \left[\frac{N!}{n!(N-n)!} \right] \quad .)$$

5. (a) Show that the second virial coefficient for the square-well potential is given by

$$B_2^{(SW)}(T) = b_0 [1 - (a^3 - 1)(e^{\beta\varepsilon^*} - 1)]$$

where

$$u_{SW}(r) = \begin{cases} \infty & r \leq \sigma \\ -\varepsilon^* & \sigma < r \leq a\sigma \\ 0 & a\sigma < r \end{cases}$$

- (b) Show that the second virial coefficient for the Sutherland potential can be written

$$B_2^{(Suth)}(T) = b_0 \left[1 - \sum_{n=1}^{\infty} \frac{1}{n!(2n-1)} \left(\frac{\varepsilon^{*n}}{kT} \right) \right]$$

where

$$u_{Suth.}(r) = \begin{cases} \infty & r \leq \sigma \\ -\varepsilon^* \left(\frac{\sigma}{r} \right)^6 & r > \sigma \end{cases}$$

(Note:

$$B_2(T) = -2\pi \int_0^{\infty} dr r^2 [e^{-\beta u(r)} - 1] \quad .)$$

6. The van der Waals equation of state is

$$(p + a\rho^2)(V - Nb) = NkT$$

where a and b are molecular parameters (constants). At the critical point of the fluid, one has

$$\left(\frac{\partial p}{\partial V}\right)_T = 0 = \left(\frac{\partial^2 p}{\partial V^2}\right)_T$$

- (a) Use these derivatives to show that the critical temperature T_c , volume V_c (for a fixed number N of molecules), and pressure p_c are given by

$$T_c = \frac{8a}{27kb} \quad V_c = 3Nb \quad p_c = \frac{9}{27b^2}$$

respectively. Also, use these results to show that $\frac{p_c V_c}{T_c} = \frac{3Nk}{8}$.

- (b) Show that the van der Waals equation of state can be re-written

$$\left(p^* + \frac{3}{V^{*2}}\right)\left(V^* - \frac{1}{3}\right) = \frac{8T^*}{3}$$

where the reduced quantities are defined by $p^* = \frac{p}{p_c}$, $V^* = \frac{V}{V_c}$, and $T^* = \frac{T}{T_c}$.

COMPREHENSIVE EXAMINATION

Department of Physics

University of Nevada, Reno

January 16, 2004; 9:00-11:30 PM

QUANTUM MECHANICS

Answer any four problems. Do not turn in solutions for more than four problems.

(1) Generalize the one-dimensional, infinite square well problem to three dimensions (= "particle in a cubic box").

(a) What is the Hamiltonian?

(b) What are the eigenvalues and eigenfunctions of the problem? Are there any degenerate eigenvalues? Are all eigenvalues degenerate?

(c) If the edge lengths of the potential box are chosen differently along the three spatial directions, does this influence the answers given under (b) and, if so, what are the new findings?

(d) For the cubic box of part (b), design a scheme to count the number of eigenvalues below a given energy E . Can your scheme be generalized to determine the number of eigenvalues between, say E and $E + \Delta E$?

(2) Consider a one-dimensional harmonic oscillator with eigenvalue equation $H|n\rangle = E_n|n\rangle$.

(a) Using raising and lowering operators, find $\Delta x \Delta p_x$ for any state $|n\rangle$.

(b) Suppose that we prepare the state $|\Psi\rangle$ of the system to be $|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$. Find $\Delta x \Delta p_x$ for this state.

(3) Consider a spin 1/2 particle of magnetic moment $\vec{M} = \gamma \vec{S}$. The spin space is spanned by the basis of the $|+\rangle$ and $|-\rangle$ states, eigenvectors of S_z with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively. At time $t = 0$, the state of the system is $|\psi(t=0)\rangle = |+\rangle$. Let the system evolve under the influence of a uniform magnetic field \vec{B}_0 , parallel to the y -axis. The Hamiltonian can then be taken to be $H = -\gamma B_0 S_y$.

(a) Calculate $|\psi(t)\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis.

(b) Find $\langle S_\alpha \rangle(t) = \langle \psi(t) | S_\alpha | \psi(t) \rangle$ for $\alpha = x, y$ or z .

(c) Using Ehrenfest's theorem find $\langle S_x \rangle(t)$, $\langle S_y \rangle(t)$ and $\langle S_z \rangle(t)$, and compare with the results from part (b).

(4) **Stationary solutions in one dimension:** A spinless particle moves in the following potential

$$V(x) = -V_0 [\delta(x) + \delta(x - a)]$$

where $\delta(x)$ indicates Dirac's delta function. V_0 is a positive constant which accounts for the strength of the attractive potentials. The final goal is to find

the possible energy eigenvalues and eigenstates for the case that the energy of the particle is less than zero: $E < 0$. The following sequence of steps may be helpful for arriving at this goal:

(a) Derive the rules for connecting a function y and its derivative y' across a delta function potential.

(b) Determine the form of physically acceptable solutions in the regions away from the delta functions.

(c) Use the results from part (a) to connect the solutions at $x = 0$ and $x = a$ and solve for the unknown coefficients.

(d) Discuss relevant special cases which you can think of.

(e) In order to discuss the symmetry of the solutions it is preferable to place the coordinate origin in the middle between the delta functions.

(f) Can you think of a real system which resembles this problem?

Note: The eigenvalue(s) of this problem can only be obtained by solving a transcendental equation. Indicate a way how to solve this equation.

(5) Vector analogy: Two given vectors which are not perpendicular upon each other span a plane in 2D. Design an algebraic procedure to replace one of the two vectors by another vector which is perpendicular upon the remaining one. What do you have to do so that the vector pair can serve as a set of orthonormal basis vectors?

Now translate the individual steps of this procedure to the case of two arbitrary kets from unitary vector space which originally are not mutually orthogonal upon each other.

Comprehensive Exam Questions

Optics/Modern Physics / Bruch

Einstein Theory of Light – Matter Interactions

- a) The atoms of a laser medium undergo repeated quantum jumps to higher levels as a result of a pumping process and then decay to a lower state converting atomic energy into light energy (photons). Describe the
- (i) $h\nu = E_2 - E_1$ (pump process)
 - (ii) spontaneous emission of a photon of energy $h\nu = E_2 - E_1$
 - (iii) stimulated emission of a photon
 - (iv) radiative de-excitation, where the atom jumps down from level 2 to the lower level 1, but no photon is emitted.
- b) In optical communication fiber optical amplifiers play an important role. Describe qualitatively the physical principle of an optical amplifier.
- c) Laser cooling is a hot topic in modern physics. Sketch some of the basic ideas governing laser cooling of atoms.

COMPREHENSIVE EXAMINATION

Department of Physics
University of Nevada, Reno
January 14, 2004; 9:00-11:30 AM

ELECTRICITY and MAGNETISM

Answer any four problems. Do not turn in solutions for more than four problems.

(1)(a) A magnetostatic field is due entirely to a localized distribution of permanent magnetization \vec{M} . Show that $\int \vec{B} \cdot \vec{H} d^3x = 0$, provided that the integral is taken over all space.

(b) If a distribution of magnetization \vec{M} gives rise to a magnetic induction field \vec{B} and if the strength of the distribution is altered to $\nu\vec{M}$ and then gives rise to a magnetic field $\nu\vec{B}$, we say there is a linear relationship between \vec{M} and \vec{B} . Use this property to show that the magnetostatic energy is given by $W = -\frac{1}{2} \int \vec{M} \cdot \vec{B} d^3x$.

(Here, the particular interest is the explanation of the factor $\frac{1}{2}$.)

(2) Given is a conducting charged sphere of radius R_1 enclosed by a dielectric shell of thickness $(R_2 - R_1)$ and dielectric constant $K = \frac{\epsilon}{\epsilon_0}$. (a) Find the fields \vec{E} and \vec{D} everywhere. Let the charge be $Q = Q_{free}$. (b) Determine the surface charge densities where they can be found. (c) Determine the potential of the conductor.

(3) A sphere of radius R , made of a linear magnetic material of permeability μ_1 is embedded in a medium of permeability μ_2 . The sphere is placed in a magnetic field \vec{H}_0 which is initially (i.e. without the sphere) uniform and pointing in the z -direction. Find the fields \vec{H} and \vec{B} inside and outside (far away from) the sphere.

(Hint: Since there are no external currents and the material is linear the use of the magnetic scalar potential is warranted for this problem.)

(4) (a) A very long (\simeq infinitely long), hollow conducting cylinder is cut into two halves lengthwise as shown. Calculate the potential both in the interior space and the exterior.

(b) How would the solution change if the constant potentials were V_1 and V_2 (instead of V and $-V$)?

(c) Calculate the surface charge density on each half.

(Hint: Symmetry requires the general solution to be of the form: $\Phi(\rho, \phi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} [a_n \rho^n \sin(n\phi + \alpha_n) + b_n \rho^{-n} \sin(n\phi + \beta_n)]$)

(5) (a) Find the \vec{E} -field associated with the scalar potential $\Phi(r, \theta) = \frac{a \cos \theta}{r^2} + \frac{b}{r}$. Here a and b are constants.

(b) What charge distribution gives rise to this scalar potential?

COMPREHENSIVE EXAMINATION

January 12, 2004;9:00-11:30 AM

Department of Physics

University of Nevada, Reno

MATHEMATICAL PHYSICS

Answer any four problems. Do not turn in (partial) solutions for more than four problems.
Each problem has the same weight.

(1) Use contour integration to evaluate the following integrals:

(a)

$$\int_0^{2\pi} \frac{d\theta}{5 - 4 \cos \theta}$$

(b)

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx$$

(c) Evaluate the following integral

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2} dx$$

so that $I(\sigma)$ has the asymptotic form of an incoming wave $e^{-i\sigma}$. Here σ is real and positive.

(2) Using standard notation, a complex function $f(z) = u + iv$ is given with

$$u(x, y) = e^{-x}(x \sin y - y \cos y) \quad \text{and}$$

$$v(x, y) = \text{unreadable!}$$

Determine $v(x, y)$ such that $f(z)$ is an analytic function. Is your expected result unique?

(3) Solve the following differential equation using the power series method (Frobenius method):

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + (x^2 + 2)y = 0.$$

(a) Determine the type of singular points if there are any.

(b) While here only one fundamental solution is asked for, what does Fuchs' theorem tell about the possibility to find the second fundamental solution?

(4) a) The matrix C is not Hermitian. Show that $C + C^\dagger$ and $i(C - C^\dagger)$ are Hermitian.

b) A and B are two noncommuting Hermitian matrices :

$$AB - BA = iC.$$

Show that C is Hermitian.