

MATHEMATICAL PHYSICS

Answer any four problems. Do not turn in (partial) solutions for more than four problems. Each problem has the same weight.

1. VECTOR CALCULUS

(a) Verify, for circular cylindrical coordinates, that

$$\vec{A} \cdot \vec{\nabla} \vec{r} = \vec{A}$$

for any vector \vec{A} . The vector \vec{r} is meant to be a general position vector and *not* to be confused with the coordinate ρ (a scalar!) indicating the distance from the z -axis.

(b) Also in circular cylindrical coordinates, evaluate the dot product $\vec{\nabla} \cdot \vec{A}$ using the definition

$$\nabla = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}.$$

(c) The vector \vec{B} is formed by the product $(\vec{\nabla}u) \times (\vec{\nabla}v)$ where u and v are scalar functions.

(i) Show that \vec{B} is solenoidal.

(ii) Show that $\vec{A} = \frac{1}{2}(u\vec{\nabla}v - v\vec{\nabla}u)$ is a vector potential for \vec{B} (i.e. $\vec{B} = \vec{\nabla} \times \vec{A}$).

2. CONTOUR INTEGRATION

Evaluate the following definite integral by contour integration:

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)^2(x^2 + 2x + 2)}.$$

3. COMPLEX FUNCTIONS

Let $A = \partial^2 w / \partial x^2$, $B = \partial^2 w / \partial x \partial y$, $C = \partial^2 w / \partial y^2$. From the calculus of real

functions of two variables $w(x, y)$, we have a saddle point if

$$B^2 - AC > 0.$$

With $f(z) = u(x, y) + iv(x, y)$, apply the Cauchy-Riemann conditions and show that neither $u(x, y)$ nor $v(x, y)$ has a maximum nor a minimum in a finite region of the complex plane.

4. MATRICES

(a) Prove, in three dimensions, that a matrix and its transpose have the same determinant.

(b) Prove that the trace of a matrix is invariant under change of basis, that is

$$\text{Tr}(A') = \text{Tr}(CAC^{-1}) = \text{Tr}(A)$$

(c) Show that the determinant of a matrix is invariant under a change of basis, i.e.

$$\det(A') = \det(A).$$

Hence, show that the determinant of a real, symmetric matrix equals the product of its eigenvalues.

(d) If the product of two matrices is zero, it is not necessary that either one be zero. In particular, show that a 2×2 matrix whose square is zero may be written in terms of just two parameters a and b , and find the general form of the matrix.

5. APPLICATION OF INTEGRAL THEOREMS

A certain force field is given by

$$\vec{F} = \hat{r} \frac{2P \cos \theta}{r^3} + \hat{\theta} \frac{P}{r^3} \sin \theta, \quad r \geq P/2$$

(in spherical polar coordinates).

(a) Examine $\nabla \times \vec{F}$ to see if a potential exists.

(b) Calculate $\oint \vec{F} \cdot d\vec{\lambda}$ for a unit circle in the plane $\theta = \pi/2$. What does this indicate about the force being conservative or nonconservative?

(c) If you believe that \vec{F} may be described by $\vec{F} = -\vec{\nabla}\psi$, find ψ . Otherwise state that no acceptable potential exists.

CLASSICAL MECHANICS

Answer all four problems.

1. The force F acting along the x -axis on a mass point with the mass m is:

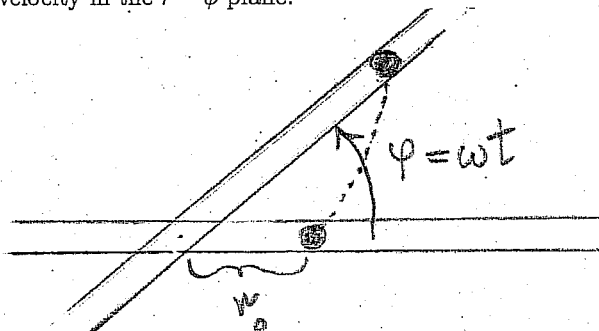
$$F = \frac{am}{v} \quad \text{where } a = \text{const. and } v \text{ is the velocity}$$

The initial conditions are:

$$x = v = 0 \quad \text{at } t = 0.$$

What is $v = v(t)$, $x = x(t)$ and $v = v(x)$?

2. A pipe placed in the $r - \phi$ plane rotates as shown with constant angular velocity in the $r - \phi$ plane.



At the time $t = 0$, a ball of mass m positioned at $r = r_0$, $\phi = 0$, placed inside the pipe is, with zero radial velocity, released from this position, moving inside the pipe without friction in the radial direction. What is the equation of the trajectory in the $r - \phi$ plane? And what is the time dependence $r = r(t)$?

3. Integrate the equations of motion for a mass point moving in the $x - y$ plane, with the mass point subject to an attractive radial force $F = F(r) = -ar$ ($a = \text{const}$), where r is the distance from the center at $x = y = 0$.
4. Write down the Lagrange function for the Kepler problem (attractive inverse square distance force law) in polar $r - \phi$ coordinates and, for this Lagrange function, the Lagrange equations of motion for the Kepler problem.

STATISTICAL MECHANICS

Please solve three of the four problems given below. Each problem has the same weight

1. Consider a rubber band of length L held at tension f . For displacements between equilibrium states

$$dE = TdS + fdL + \mu dn,$$

where μ is the chemical potential of a rubber band, and n is the mass or mole number of the rubber band. The equation of state of the rubber band is

$$E = \Theta S^2 L / n^2,$$

where Θ is a constant and L is the length of the rubber band.

- (a) What is the analog of the Gibbs-Duhem equation for the rubber band?
- (b) Calculate the chemical potential $\mu(T, L/n)$ of the rubber band.
- (c) Show that the equation of state satisfies the analog of the Gibbs-Duhem equation.

2. (a) For a system of electrons assumed non-interacting, show that the probability of finding an electron in a state with energy Δ above the chemical potential μ is the same as the probability of finding an electron absent from a state with energy Δ below μ at any given temperature T .

(b) Suppose that the density of states $D(\varepsilon)$ is given by

$$D(\varepsilon) = \begin{cases} a(\varepsilon - \varepsilon_g)^{1/2} & \text{for } \varepsilon > \varepsilon_g \\ 0 & \text{for } 0 < \varepsilon < \varepsilon_g \\ b(-\varepsilon)^{1/2} & \text{for } \varepsilon < 0, \end{cases}$$

and that at $T = 0$ all states with $\varepsilon < 0$ are occupied while the other states are empty. Now for $T > 0$, some of the states with $\varepsilon > 0$ will be occupied while some states with $\varepsilon < 0$ will be empty. If $a = b$, where is the position of μ ? For $a \neq b$, write down the equations for the determination of μ and discuss qualitatively where μ will be if $a > b$ and if $a < b$.

(c) If there is an excess of n_d electrons per unit volume than can be accommodated by the states with $\varepsilon < 0$, what is the equation for μ at $T = 0$? How will μ shift as T increases

3. A fundamental relation specifies macroscopic equilibrium states of simple systems in terms of the extensive parameters $U, V, \{N_i\}$. Example is the entropy $S = S(U, V, \{N_i\})$. The fundamental relation of a certain material is given by

$$s = s_0 + R \ln \frac{v-b}{v_0-b} + \frac{3}{2} R \ln \left\{ \sinh \left[c \left(u + \frac{a}{v} \right) \right] \right\}.$$

Here the quantities s, u, v are the "per particle" forms of the corresponding capitalized quantities. s_0, b, R, v_0, c, a are dimensioned constants.

- a) Derive the equation of state $f(P, v, T) = 0$ for this material.
- b) Show that $C_v = \frac{3}{2} \frac{R}{1 - (\frac{3}{2} R c T)^2}$.

4. A classical harmonic oscillator

$$\mathcal{H} = \frac{p^2}{2m} + \frac{Kq^2}{2},$$

is in thermal contact with a heat bath at temperature T .

- (a) Calculate the partition function for the oscillator in the canonical ensemble.
- (b) Calculate the average energy $\langle E \rangle$ of the oscillator.
- (c) Calculate the mean square deviation of the energy from the average energy $\langle (E - \langle E \rangle)^2 \rangle$.

QUANTUM THEORY

Answer any four problems. Do not turn in (partial) solutions for more than four problems. Each problem has the same weight.

1. Two identical bosons are placed in an infinite, one-dimensional square well. They interact weakly with one another via the potential

$$V(x_1, x_2) = -aV_0\delta(x_1 - x_2)$$

(where V_0 is a constant with the dimensions of energy, and a is the width of the well).

(a) First, ignoring the interaction between the particles, find the ground state and the first excited state - both the wave functions and the associated energies.

(b) Use first-order perturbation theory to estimate the effect of the particle-particle interaction on the energies of the ground state and the first excited state.

2. Born approximation:

A particle of mass m is scattered by a potential

$$V(r) = V_0 \exp\left[-\frac{r}{a}\right]$$

(a) Find the differential scattering cross-section in the first Born approximation

1. (b) The criterion of validity of the Born approximation is

$$\left| \frac{\Delta\psi^{(1)}(0)}{\psi^{(0)}(0)} \right| \ll 1$$

where $\Delta\psi^{(1)}$ is the first-order correction to the incident plane wave $\psi^{(0)}$. Evaluate this criterion explicitly for the present potential. What is the low-energy limit of your result? Is the high-energy limit less or more restrictive on the strength of the potential?

3. A particle of charge q and mass m , which is moving in a one-dimensional harmonic potential of frequency ω , is subject to a weak electric field ε .

- (a) Find the exact expression for the energy.
- (b) Calculate the energy to first nonzero correction of perturbation theory and compare it with the exact result obtained in (a).

4. Consider two non-identical particles each with angular momenta $\ell = 1$ and whose Hamiltonian is given by

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2} (\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2) \cdot \hat{\mathbf{L}}_2 + \frac{\varepsilon_2}{\hbar^2} (\hat{L}_{1z} + \hat{L}_{2z})^2,$$

where ε_1 and ε_2 are constants. Find the energy levels and their degeneracies for those states of the system whose total angular momentum is equal to 2.

5. For a particle of mass m moving in a one-dimensional box with walls at $x = 0$ and $x = L$, use the variational method to estimate:

- a) its ground state energy
- b) its first excited energy
- c) compare the results of (a) and (b) with exact solutions.

Hint: use the trial functions $\psi_0(x) = x(L-x)$ for (a) and $\psi_1(x) = x(x - \frac{L}{2})(x-L)$ for (b).

2006 Ph.D. Comprehensive Exam
Modern Physics

1. The year 2005 was designated as the *World Year of Physics*, in celebration of the 100th anniversary of publications by Albert Einstein that forever changed our understanding of the physical world. The topics of those 1905 publications are:
 - a) Energy quantization of electromagnetic radiation and the photoelectric effect.
 - b) Special theory of relativity and equivalence of mass and energy.
 - c) Statistical mechanics and the theory of Brownian motion.

Describe in general terms and comment on the significance of each of these theories or concepts. Where appropriate, give examples of modern developments or devices that are based on these theories or concepts.

[25 points each out of a total of 100 points]

2. The Nobel Prizes awarded in physics during the last ten years are the following.

2005: Glauber, Hall and Hänsch

Quantum theory of optical coherence; and laser-based precision spectroscopy, including the optical-frequency comb technique.

2004: Gross, Politzer and Wilczek

Discovery of asymptotic freedom in the theory of the strong interaction.

2003: Abrikosov, Ginzburg and Legett

Pioneering contributions to the theory of superconductors and superfluids.

2002: Davis, Koshiba and Giacconi

Pioneering contributions to astrophysics, in particular for the detection of cosmic neutrinos; and the discovery of cosmic x-ray sources.

2001: Cornell, Ketterle and Wieman

Achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for fundamental studies of the condensates.

2000: Alferov, Kroemer and Kilby

Information and communication technology; developing semiconductor heterostructures used in high-speed- and opto-electronics; and invention of the integrated circuit.

1999: t'Hooft and Veltman

Quantum structure of electroweak interactions

1998: Laughlin, Störmer and Tsui

Discovery of a new form of quantum fluid with fractionally charged excitations.

1997: Chu, Cohen-Tannoudji and Phillips

Development of methods to cool and trap atoms with laser light.

1996: Lee, Osheroff and Richardson

Discovery of superfluidity in helium-3.

In general terms, describe one of these scientific accomplishments, commenting on the significance and implications.

[25 points out of a total of 100 points]

ELECTRICITY AND MAGNETISM

Solutions

Please solve four of the five problems given below. Each problem has the same weight.

1. A very long cylinder of radius R is magnetized with the magnetization given by

$$\vec{M} = (p\rho \sin 2\phi + q\rho \cos \phi)\hat{\rho} + 2\left(\frac{1}{2}p\rho \cos 2\phi - q\rho \sin \phi\right)\hat{\phi}.$$

Here p and q are constants. Find the inside and outside potentials. (Hint: This problem can be treated using the magnetic *scalar* potential, i.e. Φ_m satisfies Laplace's equation. Why?)

2. Find the dipole moment for a spherical surface charge distribution with $\sigma = \sigma_0 \cos \theta$.

3. Charges $+q$ and $-q$ lie at the points $(x, y, z) = (a, 0, a)$ and $(-a, 0, a)$, respectively, above a grounded conducting plane at $z = 0$.

(a) Find the total force on charge $+q$.

(b) Determine the work done against the electrostatic forces in assembling this system of charges.

(c) Find the surface charge density at the point $(a, 0, 0)$.