

CLASSICAL MECHANICS

Answer all four problems.

1. The equation of motion of a mass point m along the x -axis is

$$m \frac{dv}{dt} = mf(v) \quad \text{with} \quad v = \frac{dx}{dt} \quad \text{and} \quad f(v) \quad \text{a function of } v.$$

- (a) If $x = x_0 e^{\alpha t}$ with $\alpha = \text{const.}$ and x_0 the initial position what is $f(v)$?
(b) Solve the above equation of motion

$$\frac{dv}{dt} = f(v)$$

for any function $f(v)$ in terms of definite integrals.

2. Newton's equations of motion in two dimensions and in the absence of external forces are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = 0$$

where in polar coordinates

$$x = r \cos \phi \quad \text{and} \quad y = r \sin \phi.$$

Transform the equations of motion into a reference system which is uniformly rotating around the z -axis where

$$\bar{r} = r \quad \text{and} \quad \bar{\phi} = \phi + \omega t \quad (\omega = \text{const.})$$

and split the acceleration in this reference system into the \bar{r} and $\bar{\phi}$ directions.

3. Mathematical pendulum

Obtain the equation of the mathematical pendulum with Lagrange's equation of the second kind.

4. (a) How many integrals (integrations) are needed to solve the n -body problem?
(b) How many integrals give us the conservation laws of momentum, angular momentum and energy?
(c) How many integrations are needed to solve the 1-, the 2-, and the 3-body problem?
(d) What is the relation of the conservation laws to the theorem of Noether?

ELECTRICITY and MAGNETISM

Solutions

Answer any four problems. Do not turn in (partial) solutions for more than four problems. Each problem has the same weight.

1. Find \vec{B} at a point above the center of a square current loop of edge length $2a$ carrying a current I . Compare your result to the circular loop case - a result you may remember or, otherwise, will have to reevaluate.

2. A very long cylinder of radius R is magnetized with the magnetization given by

$$\vec{M} = (p\rho \sin 2\phi + q\rho \cos \phi)\hat{\rho} + 2\left(\frac{1}{2}p\rho \cos 2\phi - q\rho \sin \phi\right)\hat{\phi}.$$

Here p and q are constants. Find the inside and outside potentials. (Hint: This problem can be treated using the magnetic *scalar* potential, i.e. Φ_m satisfies Laplace's equation. Why?)

3. The conservation of electromagnetic energy is expressed by Poynting's theorem.

- (a) Give the differential form of Poynting's theorem.
- (b) Apply Poynting's theorem to the case of a straight segment of a wire of radius a and conductivity σ_c with a potential difference V across its length ℓ and a constant current I flowing through it. Find
 - (i) Magnitude and direction of the Poynting vector \vec{S} .
 - (ii) The energy flow.
 - (iii) The energy flow as obtained from Ohm's law. Compare your result to the one above in (ii)

4. Consider two infinitely long, coaxial conductors, aligned with the z axis, and centered on the z axis. Let the inner conductor be a solid metal cylinder of radius R_1 . Let the outer conductor be an extremely thin metal film deposited on the inside of a hollow plastic cylinder of inner radius R_2 . A steady current I flows up the inner conductor (in the positive z direction) and returns down the outer conductor. The current is distributed uniformly in each conductor, respectively.
- What is the current density, $\mathbf{J}(\mathbf{r})$? (Provide both amplitude and direction. Here \mathbf{r} is the position vector, with values in the domain of all space.)
 - What is the magnetic field, $\mathbf{B}(\mathbf{r})$? (Neglect the thickness of the metal film.)
 - What is the vector potential, $\mathbf{A}(\mathbf{r})$? (Let $\mathbf{A} \rightarrow 0$ as $\mathbf{r} \rightarrow \infty$.)
5. Consider an infinite plane wave in infinite, uniform, stationary, neutral plasma. Let the plasma be composed of protons and electrons, both of number density n_0 . Superimposed on this is a density fluctuation of electrons: $n_1(\mathbf{r}, t) = n_{1e} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t])$, where \mathbf{k} is the wavevector, \mathbf{r} is the position vector, ω is the wave angular frequency, and t is time. All the electrons participate in the wave motion, oscillating according to their position with the velocity $\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_{1e} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t])$, where $\mathbf{u}_{1e} = \mathbf{k} \omega n_{1e} / (k^2 n_0)$, with $k = |\mathbf{k}|$. Let the density and velocity fluctuations of the protons be negligible in comparison to those of the electrons.
- What is the charge density, $\rho(\mathbf{r}, t)$?
 - What is the current density, $\mathbf{J}(\mathbf{r}, t)$? (Provide both amplitude and direction.)
 - Show that your results for $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ are compatible with the law of conservation of charge.
 - What is the electric field, $\mathbf{E}(\mathbf{r}, t)$?
 - What is the magnetic field, $\mathbf{B}(\mathbf{r}, t)$?

MATHEMATICAL PHYSICS

Solutions

Answer any four problems. Do not turn in (partial) solutions for more than four problems. Each problem has the same weight.

1. Evaluate the following definite integrals by contour integration:

(a) $\oint_C \frac{dz}{(z - z_0)^n}$ for $n=1,2,3$;

(b) $\int_0^\infty \frac{dx}{1 + x^4}$.

2. **Heat Flow Equation:** The upper half of the surface of a sphere of radius 1 is held at temperature 100^0 and the surface of the other half at 0^0 . Find the steady-state temperature *inside* the sphere.

3. If there is some common region in which the two complex functions $w_1 = u(x, y) + iv(x, y)$ and $w_2 = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

4. **Laurent expansion:** Expand the function $f(z) = \frac{1}{z^2 - 1}$ in a Laurent series about the point $z = 1$ valid everywhere.

5. **Exact Differential Equation:** Solve the following implicit first order differential equation:

$$(2xy + 3x^2)dx + x^2dy = 0$$

Quantum Theory

Choose 4 of 5 problems to answer.

1. Consider a system which is initially in the state

$$\psi(\theta, \varphi) = \frac{1}{\sqrt{5}} Y_{1,-1}(\theta, \varphi) + \sqrt{\frac{3}{5}} Y_{10}(\theta, \varphi) + \frac{1}{\sqrt{5}} Y_{11}(\theta, \varphi)$$

where $Y_{lm}(\theta, \varphi)$ are normalized spherical harmonics.

- Find $\langle \psi | \hat{L}_+ | \psi \rangle$;
- If \hat{L}_z were measured what values will one obtain and with what probabilities?

2. An electron is moving freely inside a one-dimensional infinite potential box with walls at $x=0$ and $x=a$. If the electron is initially in the ground state ($n=1$) of the box and we suddenly triple the size of the box (i.e., the right-hand side wall is moved instantaneously from $x=a$ to $x=3a$), calculate the probability of finding the electron in:

- the ground state of the new box;
- the first excited state of the new box.

3. Consider a one-dimensional harmonic oscillator with angular frequency ω_0 and electric charge q . At time $t=0$ the oscillator is in ground state. An electric field is applied for time τ , so the perturbation is

$$H'(t) = \begin{cases} -q\epsilon x & 0 \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

where ϵ is a field strength and x is a position operator.

- Using first-order time-dependent perturbation theory, calculate the probability of transition to the state $n=1$.
- Using the same approximation, show that a transition to $n=2$ is impossible.

Hint: Some of these expressions may be useful in solution of this problem (but may be not all of them):

$$\begin{aligned} \langle n | \hat{x} | n+1 \rangle &= \sqrt{a(n+1)} & \langle n | \hat{x} | n-1 \rangle &= \sqrt{an} \\ \langle n | \hat{x}^2 | n+2 \rangle &= a\sqrt{(n+2)(n+1)} & \langle n | \hat{x}^2 | n \rangle &= a(2n+1) & \langle n | \hat{x}^2 | n-2 \rangle &= a\sqrt{n(n-1)} \end{aligned}$$

where $a = \frac{\hbar}{2m\omega}$.

4. Use the variational method to estimate the ground state energy for the one-dimensional harmonic oscillator:

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2,$$

by choosing a trial function of the form

$$\psi(x, \alpha) = A/(x^2 + \alpha)$$

where A is determined by normalization and α is a positive real adjustable parameter. Compare the result with an exact energy and if it is not equal to it explain why.

Hint: the following integrals may be useful

$$\int_0^{+\infty} \frac{1}{(x^2 + b^2)^2} dx = \frac{\pi}{4b^3}$$

$$\int_0^{+\infty} \frac{1}{(x^2 + b^2)^3} dx = \frac{3\pi}{16b^5}$$

$$\int_0^{+\infty} \frac{1}{(x^2 + b^2)^4} dx = \frac{5\pi}{32b^7}$$

$$\int_0^{+\infty} \frac{x^2}{(x^2 + b^2)^2} dx = \frac{\pi}{4b}$$

5. Suppose we put the following perturbation in the center of the infinite square well:

$$H' = \alpha \delta(x - a/2)$$

where α is a constant.

- Find the first-order time-independent perturbation theory correction to the allowed energies. Explain why the energies are not perturbed for even n ;
- Find the second order time-independent perturbation theory correction to the ground state energy, E_1^2 .

Statistical Mechanics

Choose 4 of 5 problems to answer.

Problem 1.

A certain system has energy

$$E(S, V, N) = aN \left(\frac{N}{V} \right)^d \exp \left(\frac{dS}{Nk_B} \right),$$

where a and d are constants.

- Show that this system satisfies the ideal gas law, $PV = Nk_B T$.
- Find the specific heats C_P and C_V .
- For a closed adiabatic system, the equation of state is $PV^\gamma = \text{constant}$. Find γ .
- Show that $\gamma = C_P / C_V$. This is a general result for reversible adiabatic changes.

Problem 2.

Consider a 1-component gas of non-interacting, indistinguishable classical structureless particles of mass m at a temperature T .

- Calculate the canonical partition function as a function of number, volume and temperature.
- Find the internal energy.
- Find the pressure.
- Find the chemical potential, μ , assuming large N . What is the sign of μ in the limit of low density?

Problem 3.

Consider the following periodic one-dimensional Ising-like model for a spin-1/2 system,

$$E_N = - \sum_{i=1}^N a - J \sum_{i=1}^N s_i s_{i+1},$$

where a is a constant.

- Write the partition function as a sum of a product of exponentials. It may be convenient to define $K = \beta J$ and $C = \beta a$.
- Renormalize (decimate) the partition function by a scale factor $b=2$ (some over one half of the spins).

- (c) Assume that the partition function of part (b) can be written in the same form as that in part (a), and find recursion relations for K and C (or two variables that you may define conveniently).

Problem 4.

- a) Calculate the temperature of the surface of the Earth under the assumption that as a black body in thermal equilibrium it reradiates as much thermal radiation as it receives from the Sun. Assume also that the surface of the Earth is at a constant temperature of the day-night cycle. Use $T(\text{sun}) = 5800 \text{ K}$; $R(\text{sun}) = 7 \times 10^{10} \text{ cm}$; and the Earth-Sun distance of $1.5 \times 10^{13} \text{ cm}$.
- b) Next, consider a partially absorbant layer in the Earth's upper atmosphere which transmits the entire flux of radiation incident on it by the Sun, but which absorbs and re-emits 10% of the radiation from the surface of the Earth. Calculate the new equilibrium temperature at the surface of the Earth.

Problem 5.

Consider one mole of an ideal monatomic gas (spin zero) at 300 K and 1 atm. First let the gas expand isothermally and reversibly to twice the initial volume; second, let this be followed by an isentropic expansion from twice to four times the initial volume.

- a) How much heat (in Joules) is added to the gas in each of these two processes?
- b) What is the temperature at the end of the second process?
- c) Suppose the first process is replaced by an irreversible expansion into the vacuum, to a total volume twice the initial volume. What is the increase of entropy in the irreversible expansion, in Joules per Kelvin?